

ENABLERS OF MATHEMATICAL MODELLING: WHAT I'VE LEARNT THROUGH THREE YEARS OF ENGAGEMENT

STEVE BRODERICK

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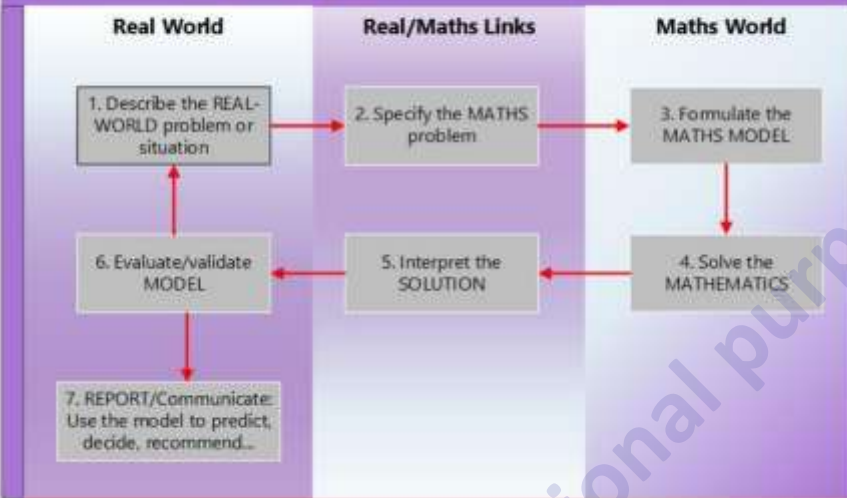
Four students have participated in the Enablers project from 2017-2019

In 2017, they were:

- Student A (Year 10)
- Student B (Year 9)
- Student C (Year 8)
- Student D (Year 8)

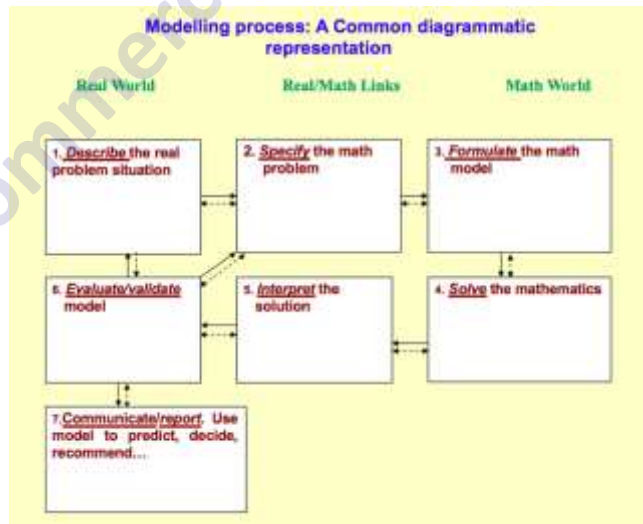
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The Mathematical Modelling Cycle

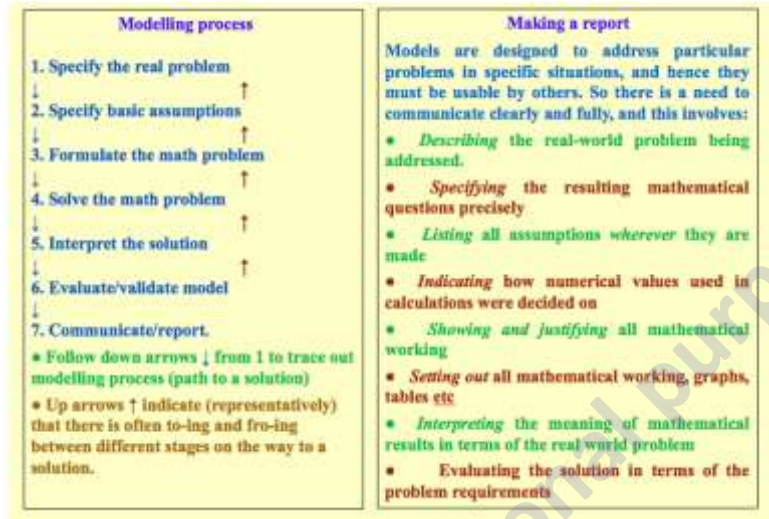


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Modelling process: A Common diagrammatic representation



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The Enablers of Mathematical Modelling

The two sets of principles – those for teachers and students – were used as a starting point during classroom observations for identifying those activities that enabled students' modelling.

The principles that relate to TEACHERS are:

- The mathematical demand of tasks does not exceed the mathematical capabilities of the student group.
- Problems are introduced so as to engage the students fully with the task context, while ensuring that goal of the task is understood.
- Assistance provided during modelling sessions (measured responsiveness) is geared to helping students use the modelling process to reach a solution, rather than treat a problem as an individual exercise.
- Students are encouraged/required to organise and report their work using headings/sections consistent with the modelling process.
- Productive forms of collaborative activity are used to enhance and hold to account the quality of on-task progress. Effective use of digital technologies. Students' interest in a problem.

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The principles that relate to STUDENTS are:

Students who model successfully*:

- ENGAGE
- WORK COLLABORATIVELY
- USE TECHNOLOGY

* It should be noted that it is possible for successful modelling to be undertaken alone, and without the use of technology.

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Teachers who implement modelling successfully in their mathematics classes:

- DEVELOP/SELECT TASKS
- ENCOURAGE STUDENTS
- RESPOND
- ANTICIPATE


*Niss, M. (2010). Modeling a crucial aspect of students' mathematical modelling. In R. Lesh, P.L. Galbraith, C.R.Haines & A. Harford, (Eds.), *Modeling Students' Mathematical Modeling Competencies: ICTMA 13* (pp. 43-50). New York, NY: Springer.

This reading can be found in the Links to Our Publications page.


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**The Enablers
Project Resources
on the website**

Classroom activities I have trialled:
www.mathsmodellingenablers.com

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Enablers website: Classroom Resources


OUR ACTIVITIES:

The activities below have been trialled in classrooms in Queensland and/or Victoria and have proven to be effective across lower, middle and upper-secondary mathematics classes. Some have been written by us, some by our project teachers in consultation with the *Enablers* team. Others have been adapted from previously published activities.

Where possible, we have provided both a worksheet and Solution Guide/Teacher Notes for each activity. The teacher notes do not always provide complete mathematical workings. They suggest possibilities for implementation and for solving problems.



Bushwalking
Worksheet




Bushwalking
Solution Guide

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Enablers website: Classroom Resources

	Is it worth the trip? Worksheet	#1	Is it worth the trip? Solution Guide
	Köchel Numbers Worksheet	#2	Köchel Numbers Solution Guide
	Sausage Sizzle Worksheet		Sausage Sizzle Solution Guide

Are worksheets that are useful for selected to different

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Enablers website: Classroom Resources

	Saving Las Arenas Worksheet	#3	Saving Las Arenas Solution Guide
	Two Second Rule Worksheet		Two Second Rule Solution Guide
	Waste Not, Want Not Worksheet	#4	Waste Not, Want Not Solution Guide

Based on compound interest type growth suited to the use of spreading techniques. An approach using spreadsheets make it suitable for Year 9-10 students; geometric series for senior.

This solution is used to illustrate how arithmetic progressions might be used in planning.

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Bushwalking Worksheet

Bushwalking Solution Guide

Survivor's Four-Flags Race

Four-Flags Worksheet

Four-Flags Solution Guide

Height of Crime

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e Modelling Enablers Project

Is it worth the trip?

The rapid change in the price of petrol has become common place in recent years. Prices also vary significantly between suburbs/towns and states. Some people have developed the habit of using apps such as the RACQ's [Fuel Finder](#) to find the best location at which to fill up. Is it simply finding the cheapest price and driving to that location effective in terms of minimising your costs?

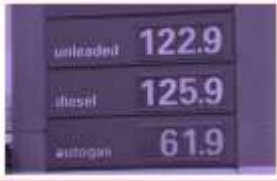
Think about your own circumstances, where you live, the type of car your family owns and any other relevant factors before developing a plan to fill up the tank of your car. This plan should include travel to and from the petrol station you select. Write up your plan as a report that includes all relevant factors, how these are included and justifications for any decisions you make.

To assist in your report, consider the following situation: Sam has just finished shopping at the 'The Gap Village Shopping Centre' and realises that the car is almost out of fuel with only about 4 litres left! Although Sam lives just across the road from the shopping centre, the car has to be returned to Jordan (Sam's sibling and the car's co-owner) with a full tank (or as near to). Sam's phone has a route finder app as

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Name of Petrol Station	Petrol Cost (Cents/Litre)
7 Eleven Albany Creek	133.7
BP Ashgrove (Waterworks Rd)	133.7
BP Stafford	133.6
Puma Eventon Park	125.7



Sam's car has the following attributes:

Make/model	Toyota Yaris Ascent Hatch Manual	Fuel Tank	42 L
Fuel Consumption	7.1L / 100km or 14.08km/L	Current Fuel Tank Level	About 4 L

Which petrol station is the best choice for Sam?

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Modelling Enablers Project

Köchel Numbers

Wolfgang Amadeus Mozart, one of the most influential composers of the Classical era, was born January 27, 1756, in Salzburg, and died December 6, 1791, in Vienna. Over the course of his lifetime, he composed more than 600 pieces of music.



Franz Xaver Köchel, Viennese librarian, lexicographer and educator, published an inclusive, chronological catalogue of Mozart's work in 1801. Köchel (K) numbers were assigned sequentially according to the date of composition. For example, Mozart's opera Die Zauberflöte was given the Köchel number 620 and was (approximately) the 620th piece of music Mozart composed. Compositions completed at the same time were listed K00, K01, and so on.

Here are some real data showing Mozart compositions, their assigned Köchel numbers, as well as the date they were completed:

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The Mc

#	Date composed	Composer	#	Date composed	Composer
10	1781	Mozart	101	1781	Mozart
11	1781	Mozart	102	1781	Mozart
12	1781	Mozart	103	1781	Mozart
13	1781	Mozart	104	1781	Mozart
14	1781	Mozart	105	1781	Mozart
15	1781	Mozart	106	1781	Mozart
16	1781	Mozart	107	1781	Mozart
17	1781	Mozart	108	1781	Mozart
18	1781	Mozart	109	1781	Mozart
19	1781	Mozart	110	1781	Mozart

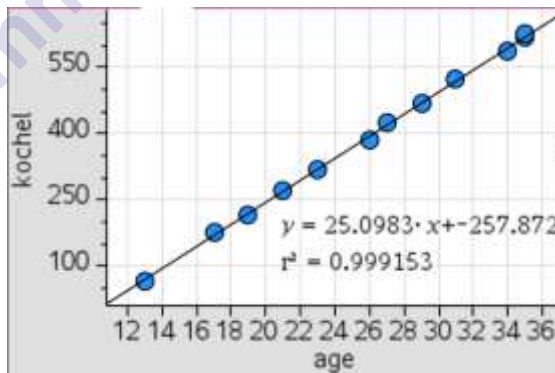
A new composition by Mozart completed in January 1791 has come to light. What Köchel number should be assigned to this new piece of music?

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5. Evaluation

Referring to the table, a composition with $K = 602$ (Jan 1781) should lie in the interval between $K = 588$ (Jan 1790) and $K = 620$ (Sept 1781). It does, so it is reasonable to infer that the model is suited to its purpose of finding approximately where on the listing of K&C numbers a newly identified composition would be placed.

On the other hand, the model was derived using only twelve points. It would be sensible to research and use more points as a starting basis, and to note and comment on any differences.

Refinement using technology

Students familiar with graphical calculator technology will likely identify the opportunity to use the regression facility to obtain the line of best fit by technical means. This is of course a legitimate approach, but one which should not be forced on those who are not familiar with the appropriate technology. (Deviation down unfamiliar technological paths has been shown to impede progress within modelling problems.) In the present case, the application of the linear regression facility gives the equation: $y = 25.25x - 275.1$. This leads to a value of $K = 608$ which is within 1% of the value calculated using eye and hand.

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Survivor's Four-Flags Race

A new game in the simulation arena. Survival involves collecting four flags. The flags are located on the four sides of a 100-metre square water tower (see figure 1). The race involves the runners starting on the western wall (collecting a flag) then racing to the other three walls collecting a flag at each wall and then back to the original starting position on the western wall. The winner is the runner who returns back at the western wall first after collecting all four flags.

There are nine competitors on the square wall each starting from a flag. Competitors are allowed the entire wall with no competitors allowed to start from the corners. Some of the competitors think that starting position is very important and a best starting position could have their chances of winning. The 7% probability experiment and 10% flag flag starting position are important. This is the runner's strategy that is the most important aspect of the race.

The grid in Figure 1 represents the 100m x 100m square for the Four-Flags Race. In Figure 1, the paths of two runners are shown. One runner (red path) starts from Point A (100, 0) and another runner (green path) is located at a different point (100, 10). For this case, assume that all runners start at the same point, so the winner is the runner who travels the shortest possible distance. Given you have determined the optimal path, discuss another starting point on the square side and determine the optimal path.

Can you generalise a solution for an $n \times n$ grid?

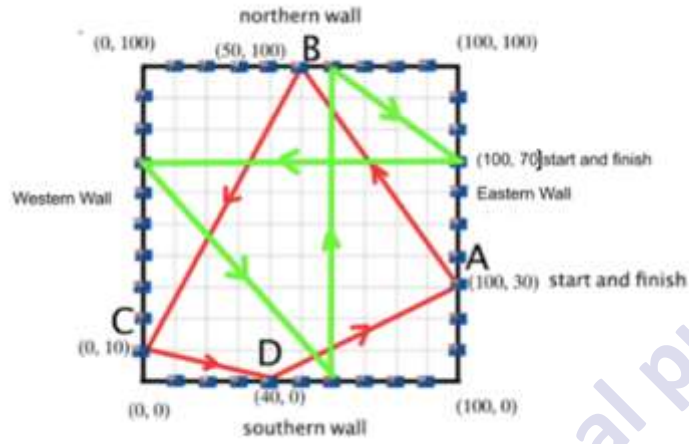


Figure 1

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$$\overrightarrow{AB} = \begin{bmatrix} 50 \\ 100 \end{bmatrix} - \begin{bmatrix} 100 \\ 30 \end{bmatrix} = \begin{bmatrix} -50 \\ 70 \end{bmatrix} \quad \therefore \quad |\overrightarrow{AB}| = \sqrt{(-50)^2 + 70^2} = 10\sqrt{74}$$

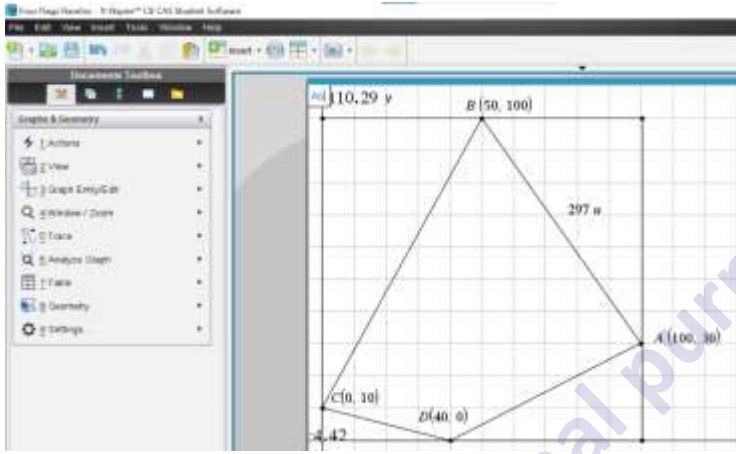
$$\overrightarrow{BC} = \begin{bmatrix} 0 \\ 10 \end{bmatrix} - \begin{bmatrix} 50 \\ 100 \end{bmatrix} = \begin{bmatrix} -50 \\ -90 \end{bmatrix} \quad \therefore \quad |\overrightarrow{BC}| = \sqrt{(-50)^2 + (-90)^2} = 10\sqrt{106}$$

$$\overrightarrow{CD} = \begin{bmatrix} 40 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 10 \end{bmatrix} = \begin{bmatrix} 40 \\ -10 \end{bmatrix} \quad \therefore \quad |\overrightarrow{CD}| = \sqrt{40^2 + (-10)^2} = 10\sqrt{17}$$

$$\overrightarrow{AD} = \begin{bmatrix} 100 \\ 30 \end{bmatrix} - \begin{bmatrix} 40 \\ 0 \end{bmatrix} = \begin{bmatrix} 60 \\ 30 \end{bmatrix} \quad \therefore \quad |\overrightarrow{AD}| = \sqrt{60^2 + 30^2} = 30\sqrt{5}$$

$$\begin{aligned} \text{Total distance} &= 10\sqrt{74} + 10\sqrt{106} + 10\sqrt{17} + 30\sqrt{5} \\ &= 297.293 \text{ m} \end{aligned}$$

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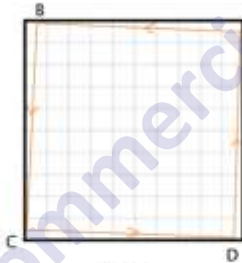


Figure 5

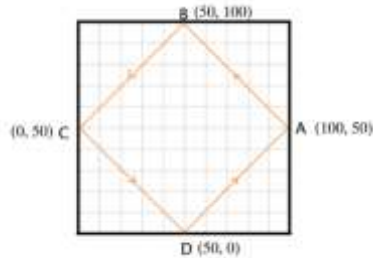
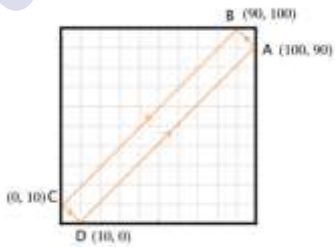


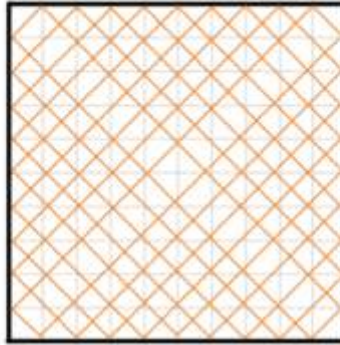
Figure 6



So the total distance is $10\sqrt{2} + 90\sqrt{2} + 10\sqrt{2} + 90\sqrt{2} = 200\sqrt{2}$, which is the same as the previous solution.

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The minimal paths for all nine runners are shown below in figure 8. All paths are equal in length which result in a minimum distance of $200\sqrt{2}$ metres.



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For a general solution, consider figure 9 below, where the minimal distance is represented by $d_1 + d_2 + d_3 + d_4$.

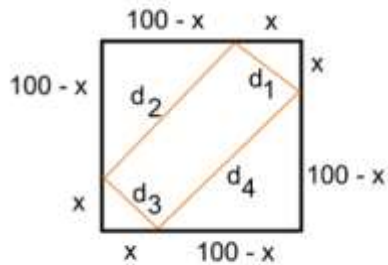


Figure 9

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$$\begin{aligned}
 d_1 &= \sqrt{x^2 + x^2} \\
 &= \sqrt{2x^2} \\
 &= \sqrt{2}x \text{ (also the same distance for } d_3\text{)}
 \end{aligned}$$

$$\begin{aligned}
 d_2 &= \sqrt{(100-x)^2 + (100-x)^2} \\
 &= \sqrt{2(100-x)^2} \\
 &= \sqrt{2}(100-x) \text{ (also the same distance for } d_4\text{)}
 \end{aligned}$$

$$\begin{aligned}
 d_1 + d_2 + d_3 + d_4 &= \sqrt{2}x + \sqrt{2}(100-x) + \sqrt{2}x + \sqrt{2}(100-x) \\
 &= 2\sqrt{2}x + 2\sqrt{2}(100-x) \\
 &= 2\sqrt{2}(x + (100-x)) \\
 &= 200\sqrt{2} \text{ metres}
 \end{aligned}$$

Which is equal to $100\sqrt{8}$ metres.

The general solution for a square of side "x" is $\sqrt{8}x$

Enablers' Project Findings

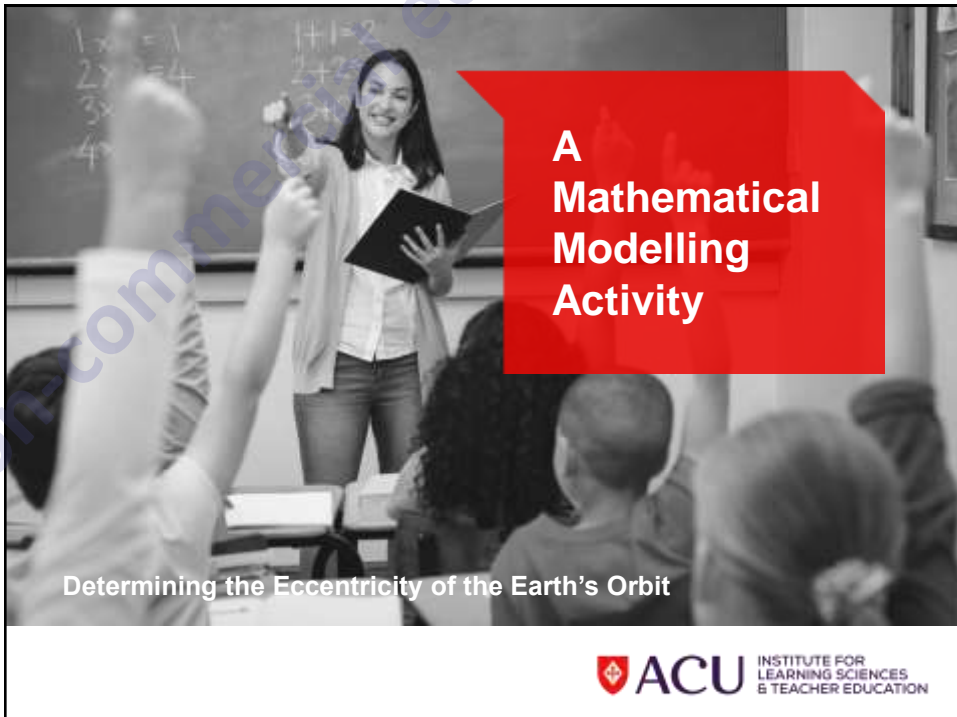
- Where possible, use real world problems/real world data v purely mathematical contrived problems
- Some students are after the 'correct answer'; modelling allows for a range of answers within the ballpark
- Using Fermi problems in the junior school allows students to become better modellers
- When using technology for curve fitting, students may aim for the best correlation coefficient which is not necessarily the best model
- Expect the unexpected

Other Resources




<https://islands.smp.uq.edu.au/>

Michael Bulmer (m.bulmer@uq.edu.au)



**A
Mathematical
Modelling
Activity**

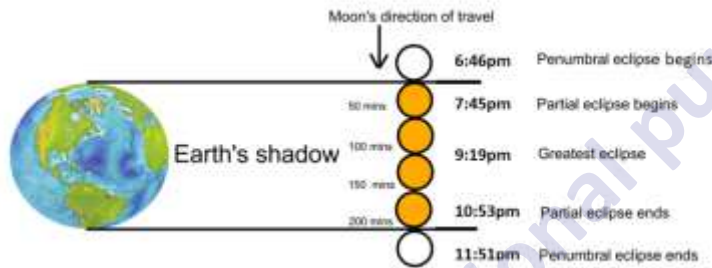
Determining the Eccentricity of the Earth's Orbit

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Times for the Lunar Eclipse

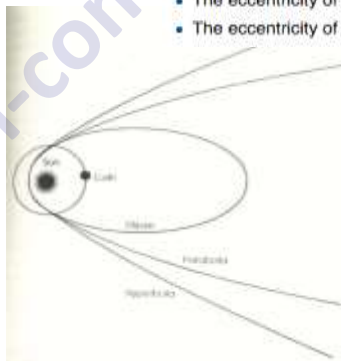
This is the first full total lunar eclipse since 2018. From Queensland, the eclipse will be seen in its entirety. Times below are in EST.

Penumbral eclipse begins	6:46pm
Partial eclipse begins	7:45pm
Total eclipse begins	9:10pm
Greatest eclipse	9:19pm
Total eclipse ends	9:28pm
Partial eclipse ends	10:53pm
Penumbral eclipse ends	11:51pm



Determining the Eccentricity of the Earth's Orbit; Kepler's Game Changing Idea

- The eccentricity of a circle is zero.
- The eccentricity of an ellipse which is not a circle is greater than zero but less than 1.
- The eccentricity of a parabola is 1.
- The eccentricity of a hyperbola is greater than 1.





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Tycho and Kepler: A STEM partnership

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Tycho & Kepler: A STEM partnership

Tycho Brahe (1546 – 1601) was the most accurate pre-telescope astronomer of his era. His data on Mars's orbit allowed Kepler to determine the elliptical nature of the orbit. He used extra-large sextants and quadrants anchored to the bedrock under his observatory to avoid wind and vibrations.



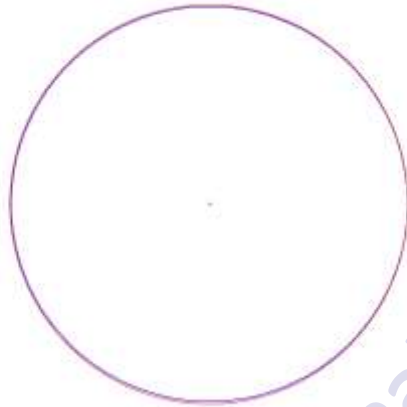
34 | Steve Broderick | Breakout 1 | Enablers Symposium, May 2021



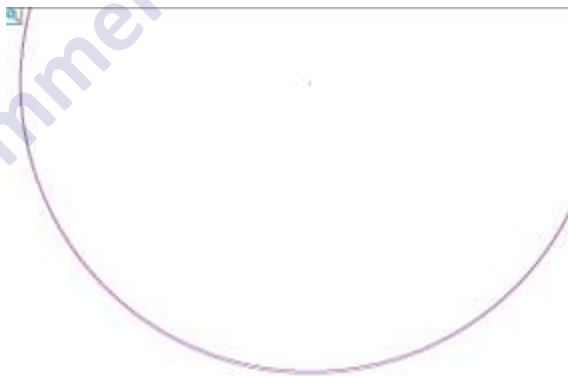
Johannes Kepler (1571 -1630) was the greatest mathematical astronomer of his day. He was totally convinced that the Sun lies at the centre of the Universe.

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Figure 3.7. The opening page from Kepler's workbooks on Mars. Courtesy of Owen Gingerich.

A section of Kepler's work from his 900-page notebook devoted to Mars.

Newton wrote:
Kepler knew ye Orb to be not circular but oval, and guest it to be Elliptical.

Kepler's guess came from a hunch actively pursued, in confrontation with all previous theories. Four years of reasoning and calculation followed his initial hunch

Kepler crossed the divide between ancient and modern astronomy and ushered in the scientific revolution.

Calculating the eccentricity of the Earth's orbit around the sun by measuring the diameter

$$e = \frac{S_p - S_a}{S_p + S_a}$$

A maths modelling activity



$$S_a = 138 \text{ mm}$$

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A maths modelling activity



$$S_p = 143 \text{ mm}$$

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A maths modelling activity

$$e = \frac{S_p - S_a}{S_p + S_a}$$

$$e = \frac{143 - 138}{143 + 138}$$

$$e = 0.0178$$

Distance of the Sun at perihelion = 152.1 million km

Distance of Sun at aphelion = 147.1 million km

$$e = \frac{d_a - d_p}{d_a + d_p}$$

$$= \frac{152.1 - 147.1}{152.1 + 147.1}$$

$$= 0.0167 \text{ (accepted value)}$$

A maths modelling activity



2017 photos



Figure 5. A SharpCap image saved as a PNG file. The prominences and active region (top left) aid in the focusing of the image.

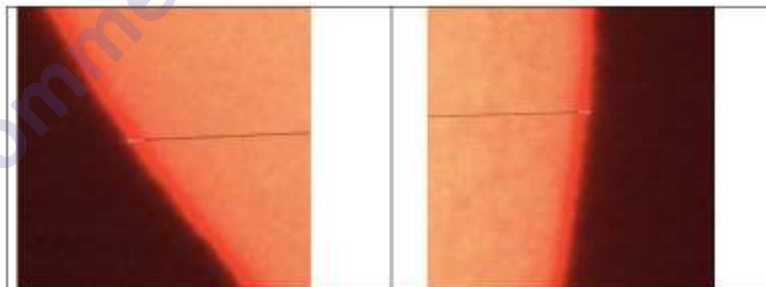
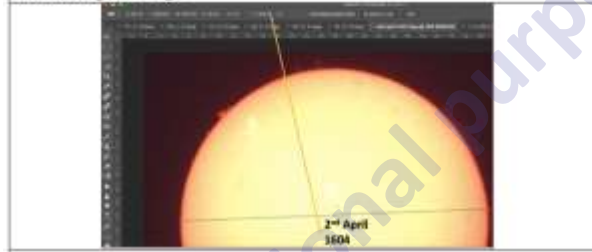
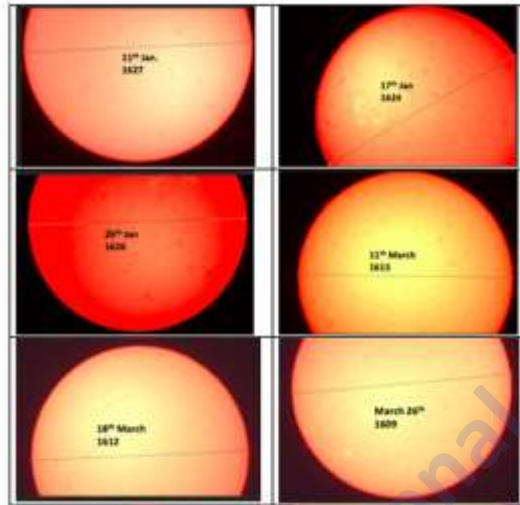


Figure 9. Enlargements of the starting point and end point for the ruler tool show the undulations in the chromosphere. The variations in length from the ruler tool due to the undulations are approximately ± 1.78 pixels.

A maths modelling activity



A maths modelling activity

Date	Day	Solar Diameter (pixels)
11 th January 2017	11	1627
17 th January 2017	17	1626
26 th January 2017	26	1624
11 th March 2017	70	1614
18 th March 2017	77	1612
26 th March 2017	85	1609
31 st March 2017	90	1605
2 nd April 2017	92	1604
7 th April 2017	97	1602
14 th April 2017	104	1599
11 th January 2018*	376*	1627*
26 th January 2018*	391*	1624*

Table 2. Solar diameter data and corresponding day number. * indicates diameters are assumed to be the same as the previous year. The error associated with the solar diameters was ± 1.78 pixels.

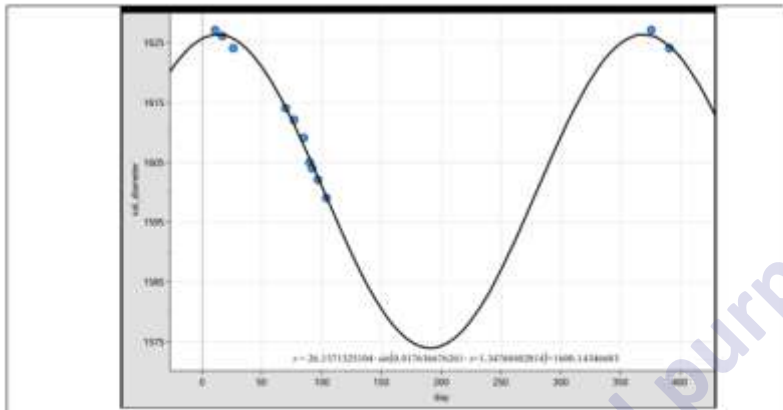


Figure 7. Represents the sinusoidal graph of day v solar diameter. The minimum and maximum represent when aphelion and perihelion occur.

$$e = \frac{1626.29 - 1573.99}{1626.29 + 1573.99}$$

$$= 0.016348$$

$$Y = 26.157 \sin(0.0176x + 1.348) + 1600.143$$

↓
↓

a
d

$$e = \frac{(d+a) - (d-a)}{d+a+d-a}$$

$$e = \frac{2a}{2d}$$

$$e = \frac{a}{d}$$

$$= \frac{26.157}{1600.143}$$

$$= 0.0163$$

A maths modelling activity

Another test of the model is determining the date for perihelion and aphelion. The model predicted the 13th day for perihelion (January 13th 2017) and the 191st day for aphelion (10th July 2017). Based on the dates in table 3, the 13th January is 8 days past the 5th January 2017 and the 10th July is 6 days past the 4th July 2017.

The percentage error is $\frac{8}{365.258} \times 100 = 2.2\%$ and $\frac{6}{365.258} \times 100 = 1.64\%$ respectively.

Perihelion	Distance (km)	Aphelion	Distance (km)
5 th January 2017	147 100 998	4 th July 2017	152 092 504
3 rd January 2018	147 097 233	7 th July 2018	152 095 566
3 rd January 2019	147 099 760	5 th July 2019	152 104 285
5 th January 2020	147 091 144	4 th July 2020	152 095 295
2 nd January 2021	147 093 163	6 th July 2021	152 100 527

Table 3. US Naval Observatory perihelion and aphelion dates and distances for 2017 to 2021.

Substituting the 2017 perihelion and aphelion distances from table 3 into the eccentricity formula:

$$e = \frac{d_a - d_p}{d_a + d_p} = \frac{152\,092\,504 - 147\,100\,998}{152\,092\,504 + 147\,100\,998} = 0.0166832 \text{ (very close to the project value of } e = 0.016345)$$

A maths modelling activity



A maths modelling activity

Solar diameter data

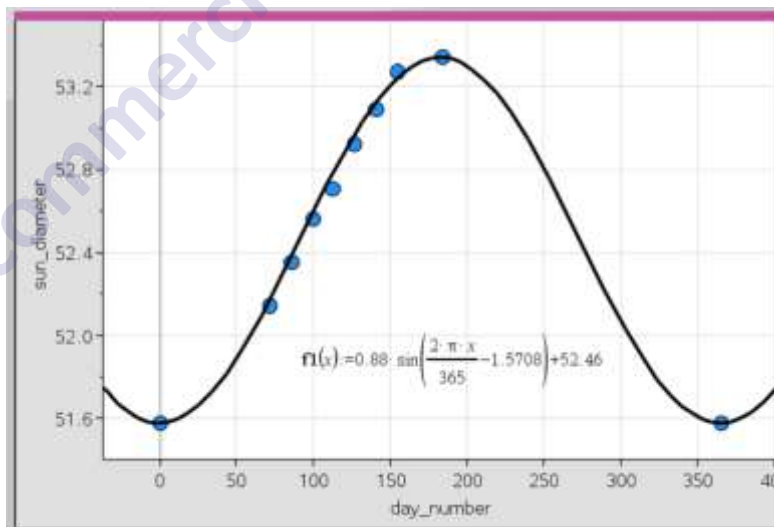
Date	Day number	Sun's diameter
4th July 2019	0	51.58
14th Sept 2019	72	52.14
29th Sept 2019	86	52.35
13th Oct 2019	100	52.56
26th Oct 2019	113	52.71
9th Nov 2019	127	52.92
23rd Nov 2019	141	53.09
7th Dec 2019	155	53.27
5th Jan 2020	184	53.34
4th July 2020	365	51.58

Model

$$Y = 0.88 \times \sin\left(\frac{2\pi x}{365} - 1.5708\right) + 52.46$$

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A maths modelling activity



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A maths modelling activity

$$e = \frac{53.34 - 51.58}{53.34 + 51.58}$$

$$= 0.01677$$

$$Y = 0.88 \sin(0.0172x - 1.5708) + 52.46$$

$$e = \frac{a}{d}$$

$$= \frac{0.88}{52.46}$$

$$= 0.01677$$

(accepted value = 0.016683, 0.52% error)

Year 11 Student D obtained an eccentricity $e = 0.01657$ which is 0.67% error.

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A maths modelling activity

Conditions			
Duration	4 weeks (including 3 hours of class time)		
Mode	Written report	Length	Up to 10 pages, maximum 2000 words, excluding appendixes
Individual/group	Individual responses	Other	
Resources available	The use of technology is required, e.g. <ul style="list-style-type: none"> ▪ computer ▪ internet ▪ spreadsheet program ▪ graphics calculator 		
Context			
You will be required to collect several images of the Sun for a particular year that has been allocated to you. (eg 2016, 2017, 2018, 2019). Data may be obtained from the link below. http://suntoday.lmsal.com/suntoday/			
<ul style="list-style-type: none"> ▪ Collect an appropriate sample of solar images to develop a model for how the diameter of the Sun varies over one full year. ▪ Use your model to predict the dates of perihelion and aphelion for your stated year. ▪ Use your data to determine the eccentricity of the Earth's orbit. 			

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A maths modelling activity

Task

After many years of painstaking calculations, Johannes Kepler (1571-1630) was able to conclude from Tycho Brahe's precise data on Mars that planets orbit the Sun in elliptical orbits. The measure of an elliptical orbit is known as the eccentricity and ranges between 0 and 1. It took several centuries to accurately determine the eccentricity of the Earth's orbit; however, today with current technologies such as the SOHO Observatory, using images of the Sun over a period of time reveal a small decrease or increase in the apparent diameter of the Sun. From these images, is it possible to accurately determine the eccentricity of the Earth's orbit?

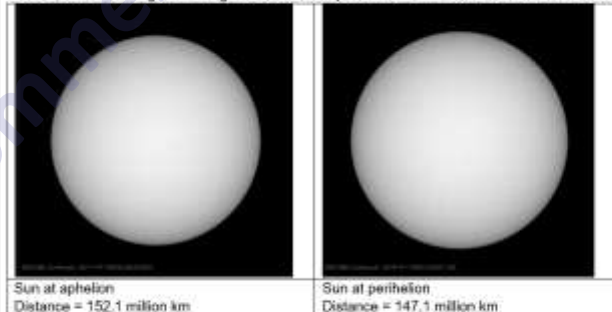
To complete this task, you must:

- use the problem-solving and mathematical modelling approach to **develop** your response
- respond with a range of understanding and skills, such as **using** mathematical language, appropriate calculations, tables of data, graphs and diagrams
- provide a response that highlights the real-life application of mathematics
- respond using a written report format that can be read and interpreted independently of the instrument task sheet
- **develop** a unique response
- use both analytic procedures and technology.

A maths modelling activity

Stimulus

Below are two images of the Sun. The image on the left was taken at perihelion when the Earth is closest to the Sun, while the image on the right was taken at aphelion when the Earth is furthest from the Sun.



To calculate the eccentricity (e), the distances at perihelion and aphelion can be used.

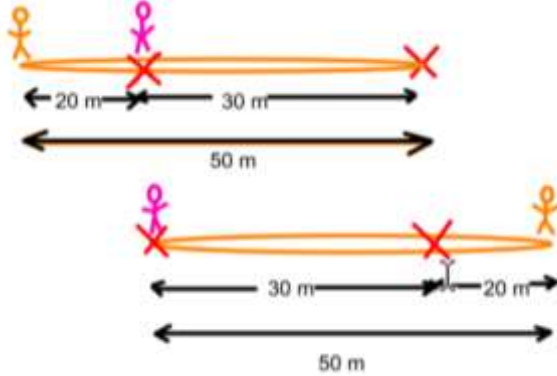
$$e = \frac{d_p - d_a}{d_p + d_a}$$

$$= \frac{152.1 - 147.1}{152.1 + 147.1}$$

$$= 0.01671$$

The accepted value is 0.0167.

A maths modelling activity



An ellipse consisting of 100 metres of string is looped around two pegs representing the two foci which were 30 metres apart. The length of the major axis ($2a$) is 70 metres.

$$e = \frac{50 - 20}{50 + 20}$$

$$= 0.43$$

A maths modelling activity



A maths modelling activity



A maths modelling activity



A maths modelling activity

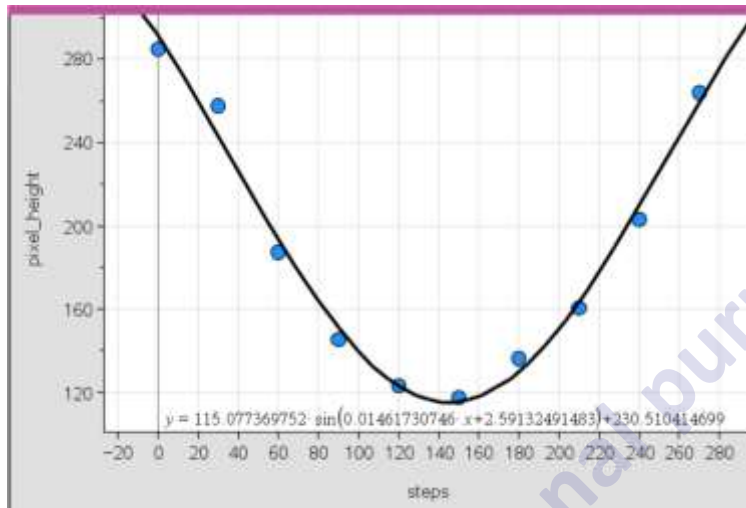


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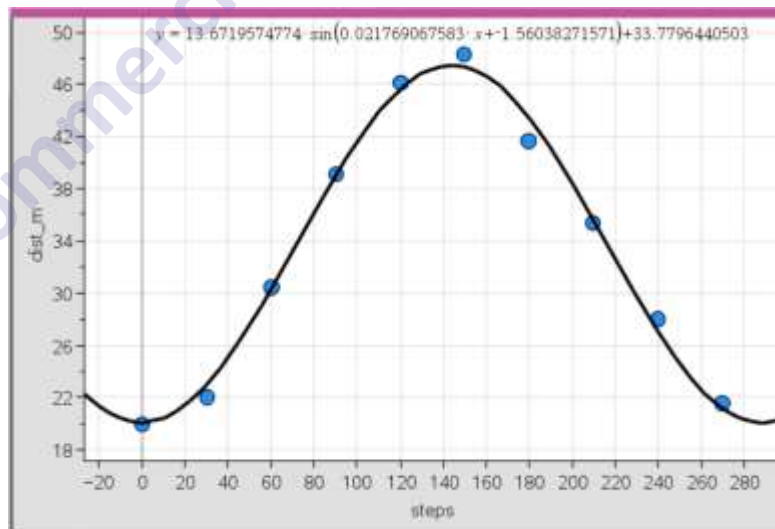
A maths modelling activity

Pixel height	Distance (metres)	Number of steps
284.41	20	0
257.7	22.07	30
187	30.42	60
145.39	39.12	90
123.29	46.14	120
117.78	48.26	150
136.57	41.65	180
160.73	35.39	210
203.06	28.01	240
263.72	21.56	270

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